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CALCULATION OF THE SLIP VELOCITY OF A RAREFIED GAS ON A SOLID SPHERICAL SURFACE WITH ALLOWANCE FOR ACCOMMODATION COEFFICIENTS

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The slip velocity of a rarefied gas with inhomogeneous temperature and mass velocity on a solid spherical surface is calculated with the use of a two-moment boundary condition in the linear approximation in terms of the Knudsen number. The dependence of the slip velocity on accommodation coefficients of the two first moments of the distribution function is studied.

Key words: *rarefied gas, slip velocity, spherical surface, two-moment boundary condition.*

Introduction. Determination of hydrodynamic boundary conditions on the surface of wetted bodies is one of the most important problems in the kinetic theory of gases [1]. Despite the large number of papers in this field, however, the problem remains open, in particular, for real surfaces. The Maxwell specular–diffuse boundary condition [1] is often used as a microscopic boundary condition imposed on the distribution function on the surfaces exposed to the gas action. All parameters of reflected molecules are determined by one quantity: diffusion coefficient. A similar situation is obtained by using the Cercignani boundary condition [2], in which all parameters of reflected molecules are determined by the accommodation coefficient of tangential momentum α_τ.

The model of Cercignani and Lampis [3] is more comprehensive. In imposing the boundary conditions, this model allows one to take into account not only the accommodation coefficient of tangential momentum α_τ but also the accommodation coefficient of the energy flux normal to the surface α_n. The use of this model of boundary conditions offers a more detailed description of the interaction processes at the gas–surface interface. A large number of problems of rarefied gas dynamics have been solved by numerical methods with the use of the Cercignani–Lampis model for boundary conditions (see, e.g., [4–6] and the bibliography therein). The use of this model of boundary conditions (as well as the Maxwell specular–diffuse model) in constructing exact analytical solutions of boundary-value problems of the kinetic theory of rarefied gases leads to insurmountable mathematical difficulties, since the problem reduces to solving inhomogeneous nonlinear integral equations, and there are no mathematical tools for constructing exact analytical solutions for these equations. At the same time, the use of the Cercignani boundary condition yields results that contradict experimental data. In particular, with the use of this boundary condition, the coefficient of thermal slipping is independent of the accommodation coefficient of tangential momentum.

The boundary condition proposed in [7] is a generalization of the Cercignani condition and allows one to take into account not only the accommodation coefficient of the first moment of the distribution function q_1, which is the accommodation coefficient of tangential momentum, but also the accommodation coefficient of the second moment of the distribution function q_2, which can be treated as the accommodation coefficient of the tangential momentum flux. With the use of the two-moment boundary conditions, the velocities of the thermal and isothermal slipping of a rarefied gas on a solid flat surface were calculated in [7]. For $q_1 = q_2 = \alpha_\tau$, the boundary condition suggested in [7] approximates the Maxwell specular–diffuse boundary condition.

With the use of the two-moment boundary condition, the slip velocity of a rarefied gas with inhomogeneous temperature and mass velocity on a solid spherical surface of radius R_0 is calculated in the present work. In [8, 9], this problem was solved for the case of an arbitrary solid smooth surface with fully diffuse reflection of gas molecules by

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the interface for Mach numbers M $\ll 1$ and, correspondingly, for limitingly low and finite Reynolds numbers Re. On the basis of the results of [8], Sone and Aoki [10] calculated the drag force in an isothermal rarefied gas flow around a sphere, the thermo- and electrophoretic forces acting on a spherical particle, and also the rate of thermophoresis of a spherical aerosol particle. A detailed review of papers published in this field can be found in [11, 12].

The Bhatnagar–Gross–Krook (BGK) model of the Boltzmann kinetic equation was used as the basic equation in the present work. The use of this model in the problem considered is caused by the fact that, having some drawbacks [13] and being the simplest model of the Boltzmann kinetic equation from the mathematical viewpoint, this model offers a correct description of the processes of slipping of a rarefied gas on a solid surface.

1. Formulation of the Problem. Derivation of the Basic Equations. We consider a rarefied gas with inhomogeneous temperature and mass velocity, which flows around a spherical surface of radius R_0 under conditions M $\ll 1$ and Re $\ll 1$. We linearize the distribution function of the gas particles with respect to the absolute Maxwellian:

$$f(\boldsymbol{r},\boldsymbol{C}) = f^0(\boldsymbol{r},\boldsymbol{C})[1+Y(\boldsymbol{r},\boldsymbol{C})].$$

Here $f^0(\boldsymbol{r},\boldsymbol{C}) = (\beta/\pi)^{3/2}\exp(-C^2)$ is the absolute Maxwellian, $\beta = m/(2k_{\rm B}T)$, $\boldsymbol{C} = \beta^{1/2}\boldsymbol{v}$ ($\boldsymbol{v}$ is the own velocity of gas molecules, $k_{\rm B}$ is the Boltzmann constant, and m is the mass of gas particles), $\boldsymbol{r} = \boldsymbol{r}_0(p/\mu_{\rm g})\beta^{1/2}$, $\boldsymbol{r}_0$ is the dimensional radius vector, $\mu_{\rm g}$ is the dynamic viscosity of the gas, and p is the static pressure; the function $Y(\boldsymbol{r},\boldsymbol{C})$ satisfies the linearized Boltzmann kinetic equation with the collision operator in the form of the BGK model. In a spherical coordinate system whose origin coincides with the particle center, this equation is written in the form

$$C_r\,\frac{\partial Y}{\partial r} + \frac{1}{r}\Big(C_\theta\,\frac{\partial Y}{\partial \theta} + \frac{C_\varphi}{\sin\theta}\,\frac{\partial Y}{\partial\varphi} + (C_\theta^2 + C_\varphi^2)\,\frac{\partial Y}{\partial C_r} + (C_\varphi^2\cot\theta - C_rC_\theta)\,\frac{\partial Y}{\partial C_\theta}$$

$$- (C_\varphi C_\theta\cot\theta + C_rC_\varphi)\,\frac{\partial Y}{\partial C_\varphi}\Big) = \beta^{-3/2}\iiint K(\boldsymbol{C},\boldsymbol{C}')\,Y(\boldsymbol{r},\boldsymbol{C}')\,d\boldsymbol{C}' - Y(\boldsymbol{r},\boldsymbol{C}), \tag{1.1}$$

$$K(\boldsymbol{C},\boldsymbol{C}') = 1 + 2\boldsymbol{C}\boldsymbol{C}' + (2/3)(C^2 - 3/2)(C'^2 - 3/2).$$

By virtue of axial symmetry of the problem, we have $\partial Y/\partial\varphi = 0$. With allowance for the boundary conditions [7], we obtain

$$Y(\boldsymbol{r},\boldsymbol{C})\Big|_S = 2C_\theta d_1 + 2C_rC_\theta d_2, \tag{1.2}$$

where the parameters d_1 and d_2 are found from the conditions

$$(1-q_1)\int\limits_{C_r<0} f(\boldsymbol{r},\boldsymbol{C})\Big|_S C_rC_\theta\,d\boldsymbol{C} = -\int\limits_{C_r>0} f(\boldsymbol{r},\boldsymbol{C})\Big|_S C_rC_\theta\,d\boldsymbol{C},$$

$$(1-q_2)\int\limits_{C_r<0} f(\boldsymbol{r},\boldsymbol{C})\Big|_S C_r^2C_\theta\,d\boldsymbol{C} = \int\limits_{C_r>0} f(\boldsymbol{r},\boldsymbol{C})\Big|_S C_r^2C_\theta\,d\boldsymbol{C}.$$

Far from the surface, $f(\boldsymbol{r},\boldsymbol{C})$ transforms to the volume distribution function written in the Barnett approximation.

We confine ourselves to low values of the Knudsen number Kn $= l/R_0 \ll 1$ (l is the mean free path of gas molecules, related to kinematic viscosity of the gas ν by the expression $\nu = l(2k_{\rm B}T/(\pi m))^{1/2}$). Following [8], we seek the solution of (1.1) in the form of the expansion into a series with respect to the parameter $k = 2\,\mathrm{Kn}/\sqrt{\pi}$:

$$Y(\boldsymbol{r},\boldsymbol{C}) = kY_1(\boldsymbol{r},\boldsymbol{C}) + k^2Y_2(\boldsymbol{r},\boldsymbol{C}) + \dots\ . \tag{1.3}$$

We substitute (1.3) into (1.1) and (1.2), taking into account that the following relations are valid in problems of gas slipping on the spherical surface ($\mu = C_r$):

$$Y_1(\boldsymbol{r},\boldsymbol{C}) = C_\theta(C_\theta^2 + C_\varphi^2 - 2)Z_0(r,\mu) + C_\theta Z_1(r,\mu),$$

$$Y_2(\boldsymbol{r},\boldsymbol{C}) = C_\theta Z_2(r,\mu) + \sum_{j=0}^{\infty} g_j(C_\theta,C_\varphi)\,\omega_j(r,\mu).$$

Here $g_j(C_\theta,C_\varphi)$ together with C_θ form the full system of orthogonal (in the sense of the scalar product) polynomials. In this case, the problem reduces to the system

$$\mu \frac{\partial Z_0}{\partial r} + Z_0(r,\mu) = 0, \qquad \mu \frac{\partial Z_1}{\partial r} + Z_1(r,\mu) = \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} Z_1(r,\tau)\exp(-\tau^2)\,d\tau,$$

$$\mu \frac{\partial Z_2}{\partial r} + Z_2(r,\mu) = \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} Z_2(r,\tau)\,\exp(-\tau^2)\,d\tau + \mu Z_1(r,\mu) - 2\frac{\partial Z_1}{\partial \mu} + 4\mu Z_0(r,\mu) - 2\frac{\partial Z_0}{\partial \mu} \tag{1.4}$$

with the boundary conditions

$$Z_1(\infty,\mu) = 2U_\theta^{(1)} + 2\mu\, S_{r\theta}^{(0)} - \Big(\mu^2 - \frac{1}{2}\Big)\frac{\partial \tau^{(0)}}{\partial \theta},$$

$$Z_2(\infty,\mu) = 2U_\theta^{(2)} + 2\mu\, S_{r\theta}^{(1)} - 2\Big(\mu^2 - \frac{1}{2}\Big)\Big(S_{r\theta}^{(0)} - \frac{\partial S_{r\theta}^{(0)}}{\partial r} - \Big(\mu - \frac{1}{2}\,\varepsilon_T\Big)\frac{\partial^2 \tau^{(0)}}{\partial r \partial \theta} + \mu\,\frac{\partial \tau^{(0)}}{\partial \theta}\Big),$$

$$Z_0(0,\mu) = 0, \quad \mu > 0, \qquad Z_0(\infty,\mu) = 0, \tag{1.5}$$

$$Z_i(0,\mu) = 2d_1^{(i)} + 2\mu d_2^{(i)}, \qquad \mu > 0, \quad i = 1,2,$$

$$S_{r\theta}^{(j)} = \frac{1}{r}\frac{\partial U_r^{(j)}}{\partial \theta} + \frac{\partial U_\theta^{(j)}}{\partial r} - \frac{U_\theta^{(j)}}{r}, \qquad j = 0,1.$$

Here, $x = r - R$; for brevity, the values of the angle θ are omitted in arguments of the functions, ε_T is the coefficient of the temperature jump, $\tau^{(0)}$ is the temperature perturbation, and $\boldsymbol{U} = \beta^{1/2}\boldsymbol{u}$ ($\boldsymbol{u}$ is the mean-mass velocity of the gas flow).

2. Basic Results. System (1.4) with the boundary conditions (1.5) was solved by the method of elementary solutions (Case method) [13]. With the use of expansion (1.3) and results obtained in [7, 14–16], the sought slip velocity of the rarefied gas on the spherical surface with allowance for accommodation coefficients of the two first moments of the distribution function is written in the form

$$U_\theta\Big|_S = kU_\theta^{(1)}\Big|_S + k^2 U_\theta^{(2)}\Big|_S + \ldots, \tag{2.1}$$

$$U_\theta^{(1)}\Big|_S = \zeta_{is} S_{r\theta}^{(0)} + \zeta_T\,\frac{\partial \tau^{(0)}}{\partial \theta}, \tag{2.2}$$

$$U_\theta^{(2)}\Big|_S = \zeta_1\, S_{r\theta}^{(1)} + \zeta_2\,\frac{\partial S_{r\theta}^{(0)}}{\partial r} + \zeta_3\, S_{r\theta}^{(0)} + \zeta_4\,\frac{\partial^2 \tau^{(0)}}{\partial r \partial \theta} + \zeta_5\,\frac{\partial \tau^{(0)}}{\partial \theta}, \tag{2.3}$$

where

$$\zeta_{is} = (2 - q_2)\,\frac{(q_1^{-1} - 1)(\sqrt{\pi} + \pi Q_1/2) - (1 - \pi/4)Q_1}{1 - \pi/4 + (1 - q_2)(1 + \pi/4 + \sqrt{\pi}Q_1)},$$

$$\zeta_T = -\frac{(2 - q_2)(1 - \pi/4)(Q_2 + 1/2)/2 - (1 - q_2)(\sqrt{\pi}Q_1/2 + \pi/4)}{1 - \pi/4 + (1 - q_2)(1 + \pi/4 + \sqrt{\pi}Q_1)},$$

$$\zeta_1 = \gamma(q_2)[\alpha(q_1) - Q_1], \qquad \zeta_2 = \gamma(q_2)[Q_2 + 0.5 + \beta(q_2)], \qquad \zeta_3 = -\gamma(q_2)[Q_2 + 1.5 + \beta(q_2)],$$

$$\zeta_4 = 0.5\gamma(q_2)[\varepsilon_n + 2\alpha(q_1) - \varepsilon_T\beta(q_2)], \qquad \zeta_5 = \gamma(q_2)\,[Q_1(Q_2 + 0.5) - \alpha(q_1)],$$

$$\alpha(q_1) = \frac{\sqrt{\pi}}{2}\,\frac{(1 - q_1)(2 + \sqrt{\pi}Q_1)}{q_1(1 - \pi/4)}, \qquad \beta(q_2) = \frac{\sqrt{\pi}}{2}\,\frac{(1 - q_2)(\sqrt{\pi} + 2Q_1)}{(2 - q_2)(1 - \pi/4)},$$

$$\gamma(q_2) = [1 + \beta(q_2)]^{-1},$$

$Q_1 = -1.01619$ and $Q_2 = -1.26632$ are Loyalka's integrals [17], $\varepsilon_T = 1.30272$, and $\varepsilon_n = -0.55844$ is the coefficient found from the condition that the gas molecules do not penetrate through the surface [18].

For $q_1 = q_2 = 1$, Eqs. (2.1)–(2.3) transform into the equations obtained in [8, 10] for the case of a spherical surface. The dependences of the coefficients ζ_i entering into (2.3) on the accommodation coefficients of tangential momentum ($q_1 = q_2 = \alpha_\tau$) are given in Table 1.

TABLE 1

q_1	q_2	ζ_1	ζ_2	ζ_3	ζ_4	ζ_5
0.25	0.25	6.44427	−2.27114	0.41919	4.6002	−3.120177
	0.50	5.41844	−1.75042	0.19327	3.7642	−2.623492
	0.75	4.43096	−1.24917	−0.02419	2.9595	−2.145376
	1.00	3.47972	−0.76632	−0.23368	2.1843	−1.684806
0.50	0.25	3.40271	−2.27114	0.41919	1.5586	−0.07861707
	0.50	2.86105	−1.75042	0.19327	1.2068	−0.06610242
	0.75	2.33964	−1.24917	−0.02419	0.86817	−0.05405563
	1.00	1.83737	−0.76632	−0.23368	0.54196	−0.04245097
0.75	0.25	2.38886	−2.27114	0.41919	0.54475	0.9352363
	0.50	2.00859	−1.75042	0.19327	0.35435	0.7863608
	0.75	1.64254	−1.24917	−0.02419	0.17106	0.6430511
	1.00	1.28992	−0.76632	−0.23368	−0.00549	0.5050008
1.00	0.25	1.88193	−2.27114	0.41919	0.03782	1.442163
	0.50	1.58236	−1.75042	0.19327	−0.07188	1.212592
	0.75	1.29398	−1.24917	−0.02419	−0.17749	0.9916044
	1.00	1.01619	−0.76632	−0.23368	−0.27922	0.7787268

TABLE 2

α_τ	ζ_{is}		ζ_T	
	Data of [6]	Data of [7]	Data of [6]	Data of [7]
0.1	17.103130	17.102710	0.2641783	0.2623022
0.2	8.224902	8.224573	0.2781510	0.2765688
0.3	5.255112	5.254859	0.2919238	0.2906146
0.4	3.762619	3.762431	0.3055019	0.3044447
0.5	2.861190	2.861055	0.3188906	0.3180640
0.6	2.255410	2.255316	0.3320949	0.3314773
0.7	1.818667	1.818608	0.3451195	0.3446892
0.8	1.487654	1.487621	0.3579692	0.3577043
0.9	1.227198	1.227184	0.3706483	0.3705269
1.0	1.016191	1.016191	0.3831612	0.3831612

With allowance for accommodation coefficients of the two first accommodation coefficients of the distribution function, we calculated the slip velocity of the rarefied gas on the solid spherical surface. The calculations of [7] show that the model of the boundary conditions for $q_1 = q_2 = \alpha_\tau$ used in that work yields results that coincide with the data obtained in [6] for the specular–diffuse model of interaction of gas particles with a flat solid surface (Table 2). The difference in coefficients of the thermal and isothermal slipping is smaller than 0.72 and 0.005%, respectively, for the entire range of α_τ.

An analysis of the measured accommodation coefficients of tangential momentum q_1 [19] shows that the values of q_1 for surfaces not subjected to special treatment (technical surfaces) are within the interval of 0.95–1.00. At the same time, no direct measurements of the accommodation coefficient q_2 were performed. However, an analysis of the measured results of the thermophoresis rate of coarse aerosol particles [20] shows that $\zeta_T = 0.3$–0.4. It follows from (2.2) that $q_2 = 0.9$–1.0 in this case.

Finally, we note that the scheme for imposing boundary conditions used above has a phenomenological character inherent in moment methods as a whole: expansion of the perturbation of the distribution function on the interface should be performed by the full system of orthogonal polynomials from molecular velocities. We cannot state *a priori*, however, accommodation coefficients of which number of moments of the distribution function should be taken into account in the description of this or that process caused by gas–surface interaction.

At the same time, the above-made comparison shows that, to correctly describe the dependence of slipping coefficients of a rarefied gas on a flat solid surface on the accommodation coefficient of tangential momentum, it suffices to take into account (and assume to be identical) the accommodation coefficients of the two first moments of the distribution function in formulation of the boundary conditions.

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REFERENCES

1. M. N. Kogan, *Rarefied Gas Dynamics. Kinetic Theory* [in Russian], Nauka, Moscow (1967).
2. C. Cercignani, “The Kramers problem for a not completely diffusing wall,” *J. Math. Phys. Appl.*, **10**, No. 3, 568–586 (1965).
3. C. Cercignani and M. Lampis, “Kinetic model for gas-surface interaction,” *Transp. Theor. Stat. Phys.*, **1**, 101–114 (1971).
4. R. G. Lord, “Some further extension of the Cercignani–Lampis gas–surface interaction model,” *Phys. Fluids*, **7**, No. 5, 1159–1161 (1995).
5. C. E. Siewert, “Generalized boundary conditions for the S-model kinetic equations basic to flow in a plane channel,” *J. Quant. Spectros. Radiat. Transfer*, No. 72, 75–88 (2000).
6. C. E. Siewert and F. Sharipov, “Model equations in rarefied gas dynamics: viscous-slip and thermal-slip coefficients,” *Phys. Fluids*, **14**, No. 12, 4123–4129 (2002).
7. A. V. Latyshev and A. A. Yushkanov, “Accommodation two-moment boundary conditions in problems of thermal and isothermal slipping,” *Inzh.-Fiz. Zh.*, **74**, No. 3, 63–69 (2001).
8. Y. Sone, “Asymptotic theory of flow of rarefied gas over a smooth boundary. 1,” *Rarefied Gas Dynamics*, **1**, 243–253 (1969).
9. Y. Sone, “Asymptotic theory of flow of rarefied gas over a smooth boundary. 2,” *Rarefied Gas Dynamics*, **2**, 737–749 (1971).
10. Y. Sone and K. Aoki, “Forces on a spherical particle in a slightly rarefied gas,” *Rarefied Gas Dynamics*, **51**, Part 1, 417–433 (1977).
11. S. A. Beresnev, V. G. Chernyak, and G. A. Fomyagin, “Motion of a spherical particle in a rarefied gas. Pt 2. Drag and thermal polarization,” *J. Fluid Mech.*, **219**, 405–421 (1990).
12. S. A. Beresnev and V. G. Chernyak, “Thermophoresis of a spherical particle in a rarefied gas: Numerical analysis based on the model kinetic equation,” *Phys. Fluids*, **7**, No. 7, 1743–1756 (1995).
13. C. Cercignani, *Mathematical Methods in Kinetic Theory*, Plenum Press, New York (1969).
14. A. V. Latyshev, V. N. Popov, and A. A. Yushkanov, “Application of the Case method in the problem on thermal slipping of a rarefied gas on a solid spherical surface,” *Sib. Zh. Indust. Mat.*, **5**, No. 3, 103–114 (2002).
15. A. V. Latyshev, V. N. Popov, and A. A. Yushkanov, “Effect of properties of the curved surface on the coefficient of isothermal slipping,” *Poverkhnost’. Rentgen, Sinkhrotron. Neitron. Issled.*, No. 6, 111–116 (2003).
16. A. V. Latyshev, V. N. Popov, and A. A. Yushkanov, “Calculation of the slip velocity of a rarefied gas, caused by inhomogeneous temperature distribution in the Knudsen layer,” *Sib. Zh. Indust. Mat.*, **6**, No. 1, 60–71 (2003).
17. S. K. Loyalka, “The Q_n and F_n integrals for the BGK model,” *Transp. Theory Stat. Phys.*, **4**, 55–65 (1975).
18. A. V. Latyshev, “Application of the Case method to solving the linearized kinetic BGK equation in the temperature-jump problem,” *Prikl. Mat. Mekh.*, **54**, No. 4, 581–586 (1990).
19. O. A. Kolenchits, *Thermal Accommodation of the Gas–Solid Systems* [in Russian], Nauka Tekhnika, Minsk (1977).
20. B. V. Deriaguin and Yu. I. Yalamov, “The theory of thermophoresis and diffusiophoresis of aerosol particles and their experimental testing,” *Int. Rev. Aerosol Phys. Chem.*, **3**, Part 2, 1–200 (1972).